The total strain energy within any annular area of inner and outer radii r0 and R is

$$\Phi = \int_0^{2\pi}\int_{r_0}^{R} \frac{1}{2}\,\sigma_{ij}\epsilon_{ij}\; r\; dr\; d\theta \;\sim\; \int_0^{2\pi}\int_{r_0}^{R} r^{(2\lambda+1)}\; dr\; d\theta.$$

Because $\Phi < \infty$ and $\lambda > -1$, the physically admissible values of λ are

$$\lambda = -\frac{1}{2},\; 0,\; \frac{1}{2},\; 1,\; \frac{3}{2},\; 2, \cdots, \frac{Z}{2},$$

where Z is and integer.

Taking $\lambda = -1/2$

$$\chi = r^{3/2} A_1 \left[\cos\frac{\theta}{2} \;+\; \frac{1}{3}\;\cos\frac{3\theta}{2}\right] + \bigcirc\left(r^2\right) + \bigcirc\left(r^{5/2}\right) + \cdots,$$

$$\sigma_{ij} = A_1 r^{-1/2}\;\tilde{\sigma}^{\mathrm{I}}_{ij}(\theta) \;+\; \bigcirc_{ij}\left(r^0\right) + \bigcirc_{ij}\left(r^{1/2}\right) + \;\cdots\;.$$

Rewriting

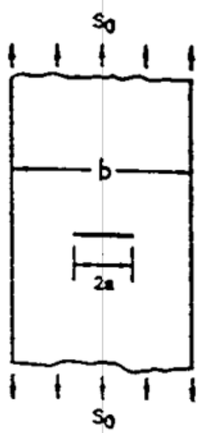
$$A_1 = K_{\mathrm{I}}/\sqrt{2\pi},$$

$$\sigma_{ij} = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}}\,\tilde{\sigma}^{\mathrm{I}}_{ij}(\theta) \;+\; T\delta_{ix}\delta_{jx}$$
$$+\;\text{(terms which vanish at crack tip)},$$

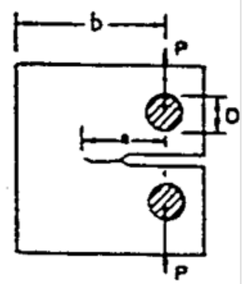
K_I is the stress intensity factor for mode I. σ_{ij} is the Kronecker delta.

Possible Crack Configurations (I)

SS01



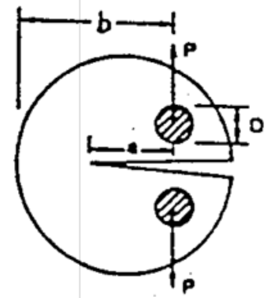
SS02



B = THK

S_0=P/BD

SS03



B = THK

S_0 = P/BD

Real Cracks in Real Structures

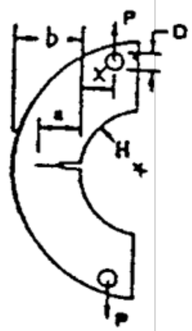
- Most structures and components contain surface cracks
- Surface cracks may be idealized as

  -- quarter elliptic cracks/corner cracks
  -- semi-elliptical/thumbnail cracks
  -- elliptical/embedded cracks

- Cracks may be present at holes, pins or other stress concentrators
- Cracks may be subjected to combinations of bending, tension, torsion
- Cracks may also have multiple crack combinations
- Simple fracture mechanics approach applicable in many cases

Possible Crack Configurations (II)

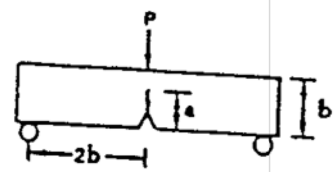
SS04



B = THK

$S_0 = P/BD$

SS05

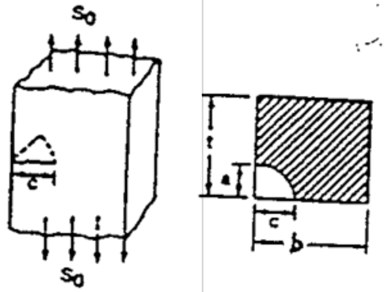


B = THK

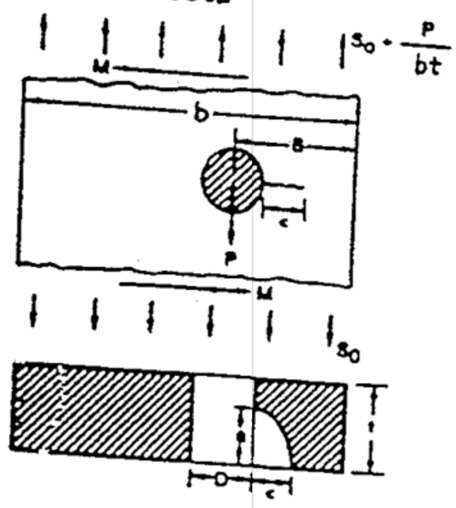
$S_0 = \frac{6P}{B\,b}$

Possible Crack Configurations (III)

CC01

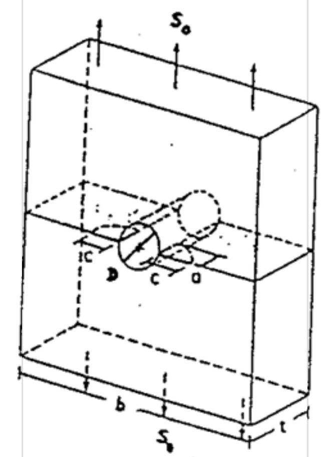


CC02



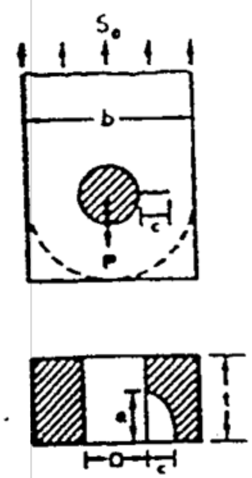
$S_1 = P/Dt \quad S_2 = \frac{6M}{bt^2}$

CC03



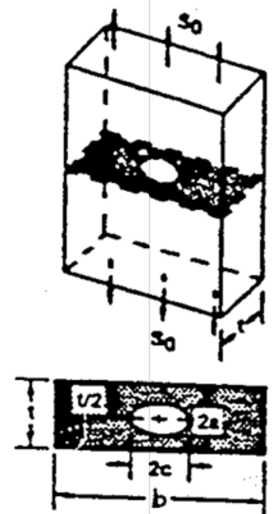
Possible Crack Configurations (IV)

CC04



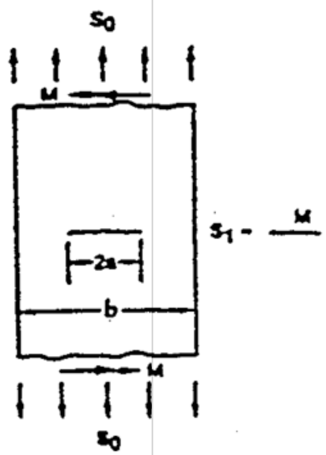
$S_0 = P/Dt$

EC01

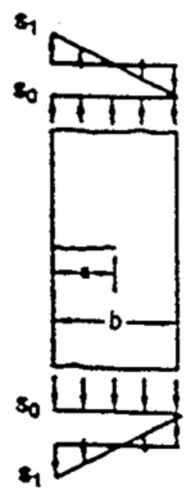


Possible Crack Configurations (V)

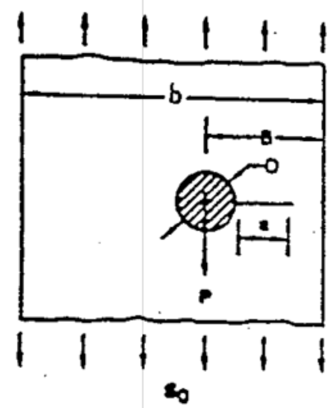
TC01



TC02

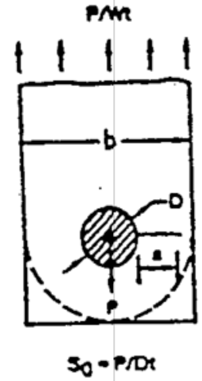


TC03

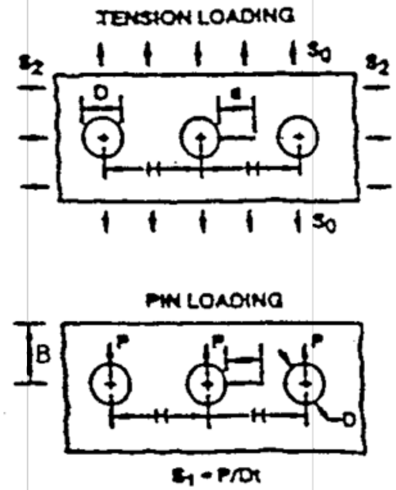


Possible Crack Configurations (VI)

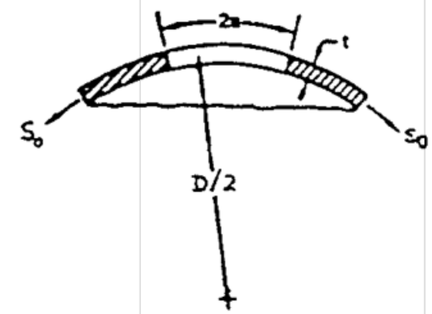
TC04



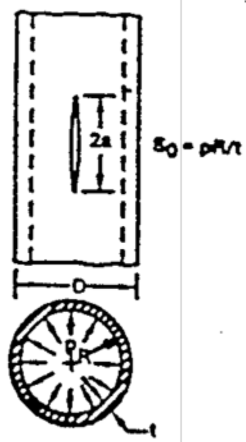
TC05

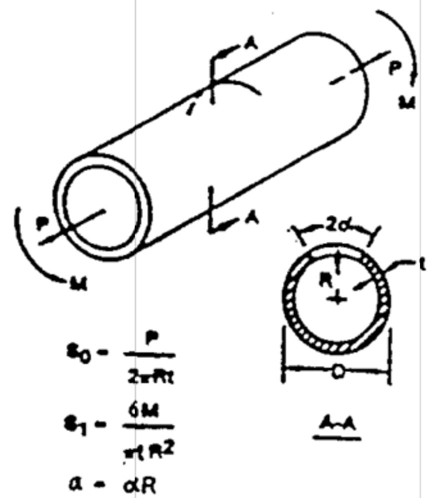


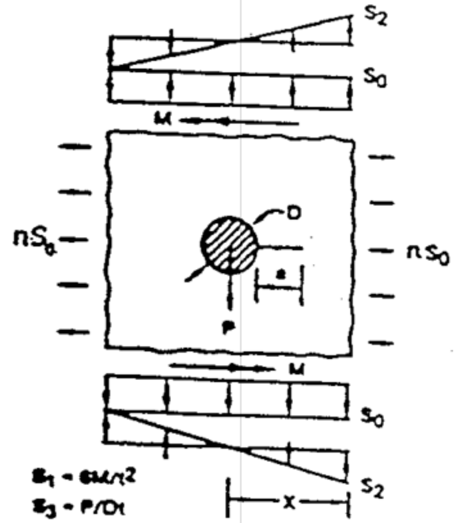
TC06



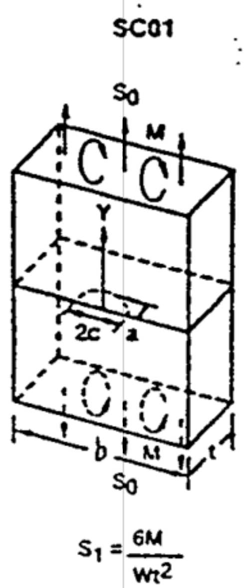
Possible Crack Configurations (VII)

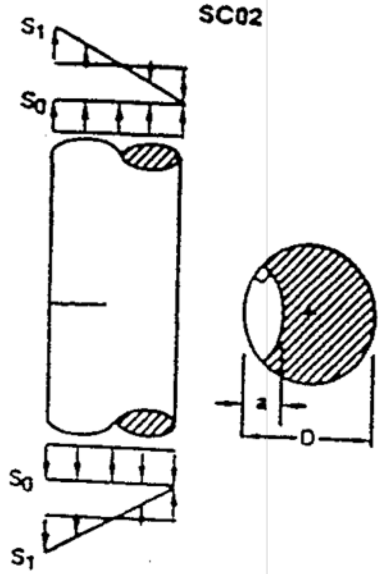


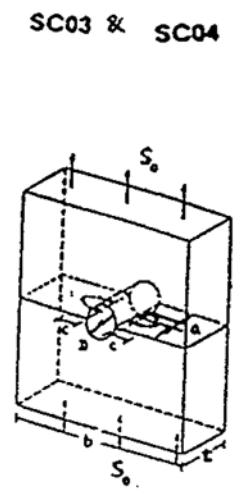




Possible Crack Configurations (VIII)

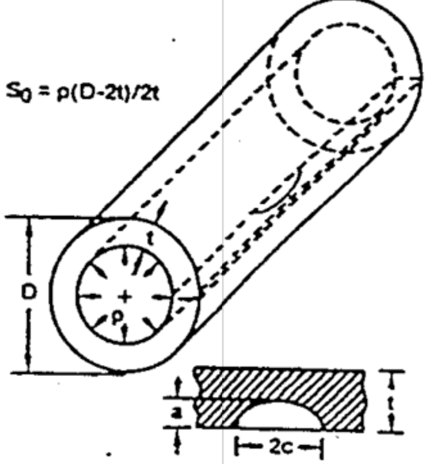




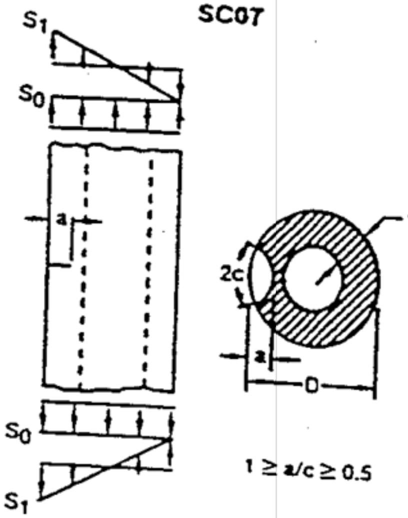


Possible Crack Configurations (IX)

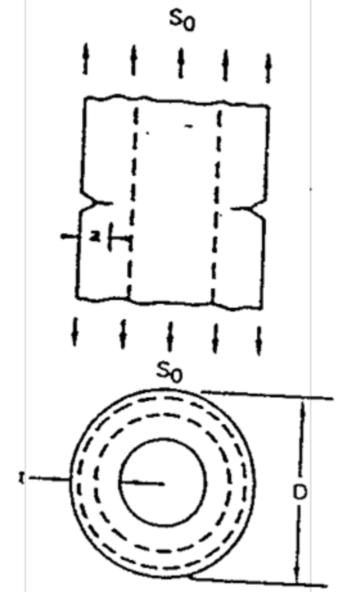
SC05 & SC06



SC07

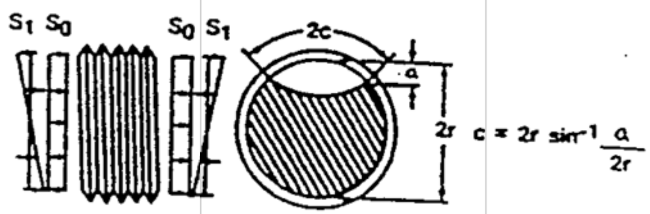


SC08

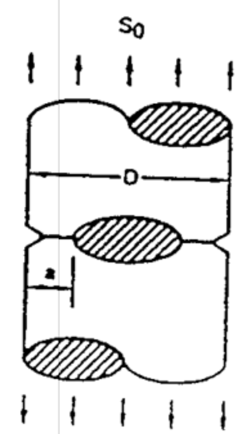


Possible Crack Configurations (X)

SC09



SC10



Monotonic Plastic Zone Sizes

- Size depends on stress state
- Irwin (1966) showed that:
  - PZS = (A/3*PI)*[DK/s_y]
  - A = 3 (plane strain)
  - A = 1 (plane stress)
- Other expressions by Dugdale and Barrenblatt

Cyclic Plastic Zone

- Smaller than monotonic plastic zone
- Affected by reversed plasticity
- PZS = (1/PI)*{DK/2s_y] (Rice, 1967)
- Residual wake left behind crack-tip
- Residual stresses induced as a result
- May promote crack closure

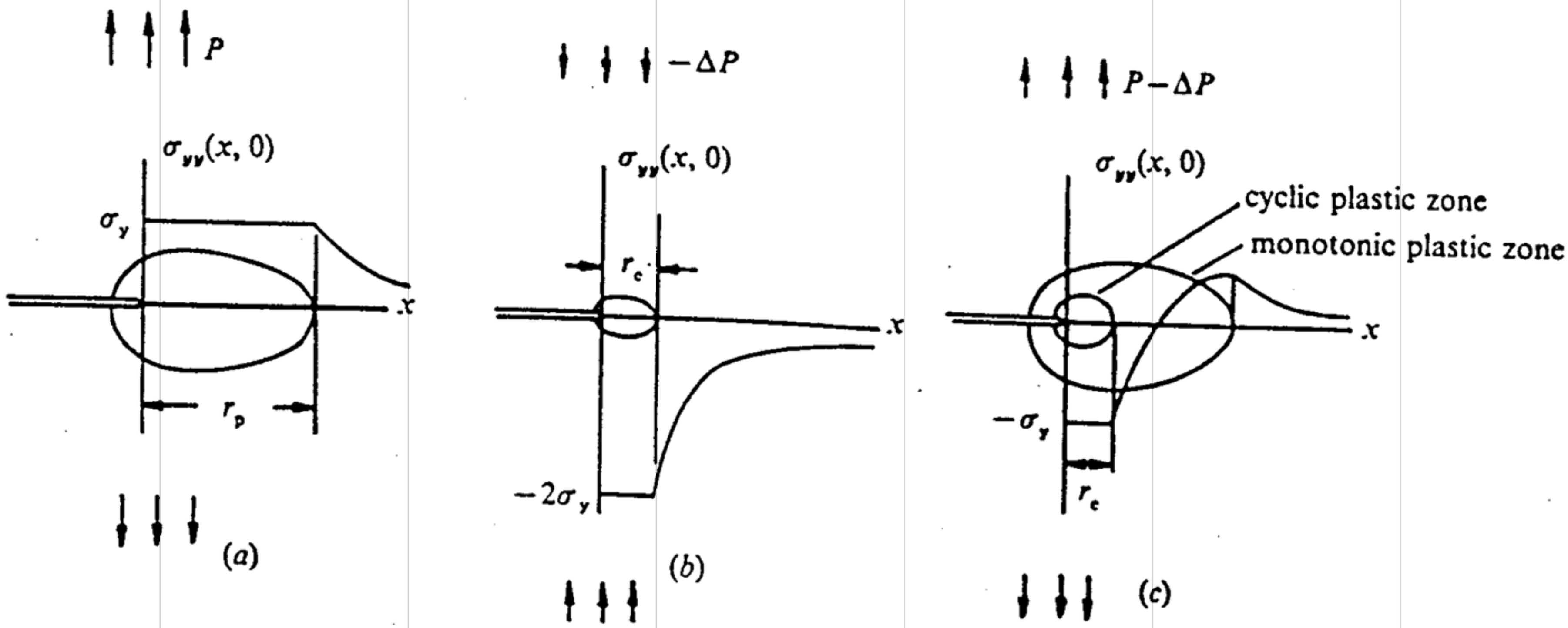
Crack Opening Displacement

- Change in CTOD between max and min load
- Function of delta K, E and yield stress
- Crack extension related to this change
- Crack opens to maximum load
- Extension due to plastic processes

Plastic Zone Size Under Cyclic Loading

- PZ existence under cyclic loading long recognized
- Paris (1960), McClintock (1963) & Rice (1967)
- Experimental evidence by Hahn, Hoagland & Rosenfield (1972)
- PZ controled by reversed plasticity
- PZ much smaller under cyclic loading

Schematic representation of the development of cyclic plastic zone upon unloading (After Rice, 1967)



$$r_c \approx \frac{1}{\pi}\left(\frac{\Delta K_I}{2\sigma_y}\right)^2.$$

(a) Monotonic plastic zone created by a far-field load P.
(b) Stress distribution due to the reduction of the reduction of the load by ΔP which, when superimposed with (a), givethe result in (c).

Modeling of Plastic Zone Under Cyclic Loading

- For proportional plastic flow monotonic eqns used
- Upon load reversal, P is reduced to P - DP
- Loading parameter replaced by DP
- s_y replaced by $2s_y$ in monotonic eqns for PZ
- For elastic perfectly plastic solid s in PZ = $-s_y$
- For plane stress conditions: $r_c = (1/PI)(DK/2s_y)^2$

Consequences of Reversed Plastic Flow

- Residual plasticity remains even after unloading
- Reisdual plasticity has implications for VA loading
- Residual stresses self equilibrating
- -ve tip stresses offset by +ve stresses ahead of tip
- Non closing compressive loading induced residual tension
- Cyclic variation induces change in CTOD

Dugdale Model - Estimates of PZ for Mode I Crack (1960)

- Thin plastic strip of elastic perfectly plastic solid
- Plastic zone loaded by s_y over r_p
- $r_p = (PI/8)*(K_I/s_y)^2$
- Similar to Irwin's estimates for plane stress
- CTOD is consequence of necking ahead of crack

The Dugdale Model

The size of the yield zone ahead of a mode I crack in a thin plate of an elastic-perfectly plastic solid (subject to plane stress deformation) was estimated by Dugdale (1960). If the traction $\sigma_{yy} = \sigma_x$ were to be applied simultaneously along the length of the strip $a < |x| < a + r_p$, it would superimpose a negative stress intensity factor K_I'' on K_I', where

$$K_I'' = -\sigma_y \sqrt{\pi(a + r_p)} + 2\sigma_y \sqrt{\frac{(a + r_p)}{\pi}} \sin^{-1}\left(\frac{a}{a + r_p}\right) \qquad (1)$$

Since a singular, deviatoric stress state cannot exist at the boundary of the plastic zone, $K_I' + K_I'' = 0$ (see Hellan, 1984, for further details). Solving for r_p, one finds that

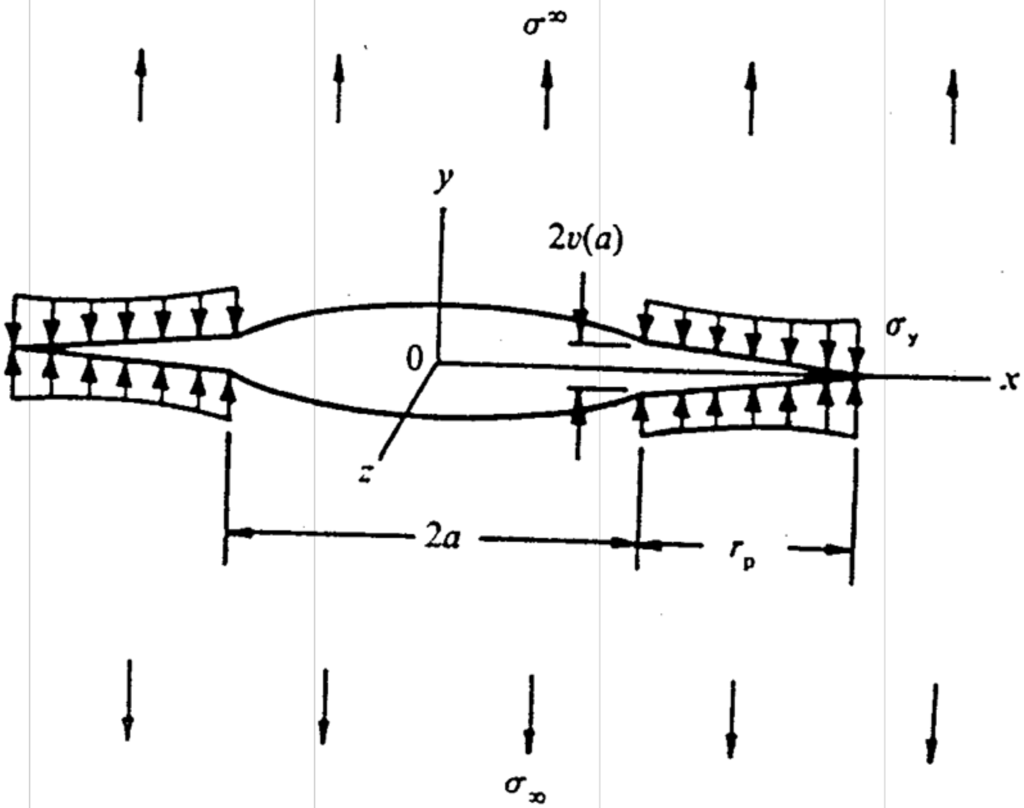
$$\frac{r_p}{a} = \sec\left(\frac{\pi\sigma^\infty}{2\sigma_y}\right) - 1 \qquad (2)$$

For $\sigma^\infty << \sigma_y$ and hence for $r_p << a$, Eq. (1) will asymptotically lead to a plastic zone size

$$r_p = \frac{\pi}{8}\left(\frac{K_I}{\sigma_y}\right)^2$$

This asymptotically exact result due to Dugdale compares well with the Irwin approximation for plane stress.

A schematic representation of the Dugdale plastic zone model



Barenblatt Model (1962)

- Analogue to strip yield model for brittle materials
- Consider s_{yy} = bond rupture strength (E/10)
- Critical crack size f(crack-tip cohesive zone)
- Or critical crack size = f(COD) - (Rice, 1968)

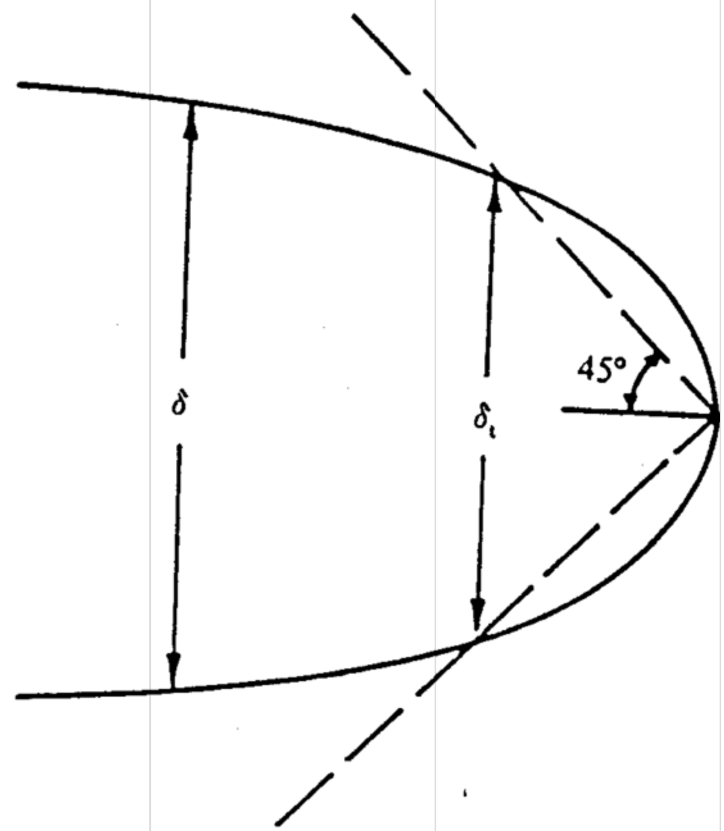
Elastic-Plastic Fracture Mechanics

- LEFM valid only for limited plasticity
- EPFM needed for many cases
- CTOD (Wells, 1963)
- J integral (Rice 1968)
- Delta J of Delta(CTOD)

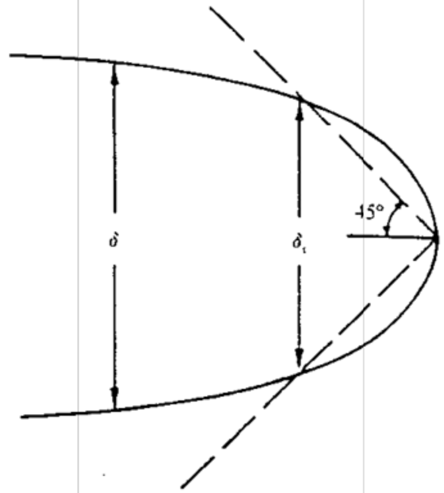
Crack-Tip Opening Displacement

- CTOD expressions derived from Dugdale model
- More accurate models include effect of hardening
- CTOD definition somewhat arbitrary
- Fracture occurs at critical CTOD
- FCG related to delta(CTOD)

Definition of crack tip opening displacement, δt



Crack Tip Opening Displacement



The definition of δt is somewhat arbitrary because the distance between the crack faces, $\delta = \mu_y(x,0^+) - \mu_y(x,0^-)$ varies as $(-x)^{1/(n+1)}$ as the crack tip is approached.

A commonly used operational definition of δt is based on the distance between two points on the upper and lower crack faces where two 45°C lines drawn from the deformed crack tip intercept the crack faces.

$$\delta_t = d_n \frac{J}{\sigma_y}$$

where d_n is a function of α, ε_y and n. d_n ranges in value from about 0.3 to 0.8 as n is varied from 3 to 13.

J Integral and Conditions of J-Dominance

- $$J = \int_\Gamma \left(\omega dy - T \cdot \frac{\partial u}{\partial x} ds \right)$$

  where u = displacement vector, y = direction along normal to crack plane, s = arc length, T = traction vector, ω = strain energy density,

  $$\sigma_{ij} = \partial \omega / \partial \varepsilon_{ij}$$

- **For linear elastic and non-linear elastic behavior - J path independent**

- **Rice (1968) showed that**

  $$J = G = -\frac{\partial (PE)}{\partial a}$$

- **Hutchinson (1983) - Jvalid when:**
  -- J_2 deformation theory of plasticity gives adequate model of $\sigma - \varepsilon$ behavior
  -- Damage and high strain region within HRR field

Hutchinson-Rice-Rosengreen Singular Fields

- Hutchinson (1968) & Rice and Rosenfield (1968)
- Elastic power law material
- Ramberg-Osgood relationship characterizes matrix behavior
- J_2 Deformation Theory
- J is a measure of the intensity of the cracktip fields
- $J = J_c$ when conditions of J dominance satisfied (ASTM E813)

Hutchinson-Rice-Rosengreen (HRR) Singular Fields

- **Developed for non-linear elastic solids - small strain monotonic deformation**
  - -- **Hutchinson (1968)**
  - -- **Rice & Rosegreen (1968)**

- **Near-tip fields given by**

$$\sigma_{ij} = \sigma_y \left(\frac{J}{\alpha \sigma_y \varepsilon_y I_n r} \right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(\theta, n)$$

$$\varepsilon_{ij} = \alpha \varepsilon_y \left(\frac{J}{\alpha \sigma_y \varepsilon_y I_n r} \right)^{\frac{n}{n+1}} \tilde{\varepsilon}_{ij}(\theta, n)$$

$$u_i = \alpha \varepsilon_y \left(\frac{J}{\alpha \sigma_y \varepsilon_y I_n} \right)^{\frac{n}{n+1}} r^{1/(n+1)} \tilde{u}_i(\theta, n)$$

$\tilde{\sigma}_{ij}(\theta, n)$, $\tilde{\varepsilon}_{ij}(\theta, n)$ **and** $\tilde{u}_i(\theta, n)$ **are universal functions.**

Conditions for J Dominance

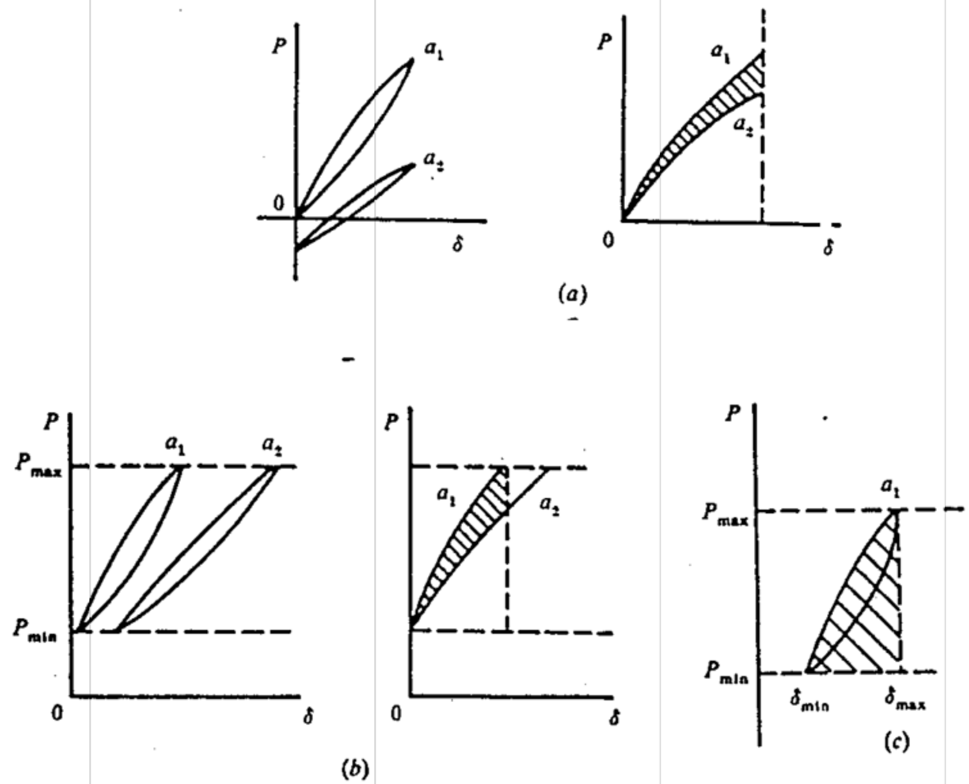
- See review article by Hutchinson (1983)
- Deformation theory of plasticity must be adequate
- This is true for proportional loading (monotonic loading)
- J_2 theory not satisfied for elastic-power law plastic solid
- Region of finite strains within process zone
- Finite strain effects significant over 3*CTOD

Condition For J Dominance

For J-controlled crack growth, Hutchinson & Paris (1979) have suggested that the regime of elastic unloading and nonproportional loading should be confined to well within the zone of J-dominance. In other words,

$$\frac{dJ}{da} >> \frac{J}{R} \text{ and } \Delta a << R$$

Determination of J-integral with stabilized cyclic hysteresis loops.



(a) Hysteresis loops for two different crack lengths in displacement-controlled fatigue and the translation of the rising part of the stabilized hysteresis loop to a common origin. (b) Similar method for load-controlled fatigue with the minimum load being employed as the reference point. (c) Determination of J using a single specimen.

Fracture Processes and J Dominance

- Region of J dominance must engulf fracture process zone
- HRR solutions hold over 20-25% of PZS in ductile solids
- J dominance specimen dependent for large scale yielding
  - 1% of length of uncracked ligament for CCT panel
  - 7% for deeply noched bend bar (McMeeking & Parks, 1979)
- R > GS for intergranular/transgranular fracture
- R > particle spacing for ductile dimpled fracture